

# A note on conjectures of F. Galvin and R. Rado

François G. Dorais

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## Abstract

In 1968, Galvin conjectured that an uncountable poset  $P$  is the union of countably many chains if and only if this is true for every subposet  $Q \subseteq P$  with size  $\aleph_1$ . In 1981, Rado formulated a similar conjecture that an uncountable interval graph  $G$  is countably chromatic if and only if this is true for every induced subgraph  $H \subseteq G$  with size  $\aleph_1$ . Todorćević has shown that Rado's Conjecture is consistent relative to the existence of a supercompact cardinal, while the consistency of Galvin's Conjecture remains open. In this paper, we survey and collect a variety of results related to these two conjectures. We also show that the extension of Rado's conjecture to the class of all chordal graphs is relatively consistent with the existence of a supercompact cardinal.

## 1 Introduction

Throughout the following,  $G$  will denote a (simple loopless) graph with vertex set  $V = V_G$  and edge relation  $E = E_G$ . For a set  $X \subseteq V$ ,  $G_X$  denotes the induced subgraph with vertex set  $X$ . A clique of  $G$  is a set  $X \subseteq V$  such that  $G_X$  is the complete graph on  $X$ . Dually, an anticlique is a set  $X \subseteq V$  such that  $G_X$  is the empty graph on  $X$ . The conjectures of Galvin and Rado concern equalities between certain cardinal characteristics in certain classes of graphs. These cardinal characteristics are the following.

- The **clique number** is

$$\omega(G) = \sup \{|X| : X \text{ is a clique of } G\}.$$

- The **stability number** is

$$\alpha(G) = \sup \{|X| : X \text{ is an anticlique of } G\}.$$

- The **chromatic number**  $\chi(G)$  is the smallest size of a cover of  $V$  by anticliques.
- The **clique-cover number**  $\theta(G)$  is the smallest size of a cover of  $V$  by cliques.

Clearly,  $\omega(G) \leq \chi(G)$  and  $\alpha(G) \leq \theta(G)$ . In view of this, it is natural to ask when the equalities  $\omega(G) = \chi(G)$  and  $\alpha(G) = \theta(G)$  hold. It is easy to check that both equalities fail for the odd cycle  $C_{2n+1}$  when  $n \geq 2$ . In 1960, Berge conjectured that the minimal finite graphs for which these equalities fail are precisely the odd cycles  $C_{2n+1}$  and their complements  $\overline{C}_{2n+1}$ , for  $n \geq 2$ . This fact, the Strong Perfect Graph Theorem, was established by Chudovsky, Robertson, Seymour, and Thomas in 2002.

Thus, if  $G$  is a finite graph that contains no induced copies of the odd cycle  $C_{2n+1}$  of length  $2n + 1$  nor its complement  $\overline{C}_{2n+1}$  for  $n \geq 2$ , then the equalities  $\omega(G) = \chi(G)$  and  $\alpha(G) = \theta(G)$  hold not only for  $G$ , but also every induced subgraph of  $G$ . In fact, we see that the first equality holds for every induced subgraph of  $G$  if and only if the second equality holds for every induced subgraph of  $G$ . This celebrated equivalence, the Perfect Graph Theorem, was also conjectured by Berge in 1960, and proved by Lovász in 1972.

**Theorem 1.1** (Lovász [9, 10]; Chudnovsky–Robertson–Seymour–Thomas [3]).  
*The following are equivalent for every graph  $G$ .*

- (a)  $G$  contains no induced copies of the odd cycle  $C_{2n+1}$  nor its complement  $\overline{C}_{2n+1}$  for  $n \geq 2$ .
- (b)  $\omega(G_X) = \chi(G_X)$  for every finite  $X \subseteq V$ .
- (c)  $\alpha(G_X) = \theta(G_X)$  for every finite  $X \subseteq V$ .
- (d)  $\alpha(G_X)\omega(G_X) \geq |X|$  for every finite  $X \subseteq V$ .

Where  $G_X$  denotes the induced subgraph of  $G$  with vertex set  $X$ .

A graph  $G$  that satisfies all of these equivalent properties is known as a **perfect graph**.

Several common types of graphs are known to be perfect. The first to be identified as such is probably the class of comparability graphs. Recall that a graph is a **comparability graph** if it has a transitive orientation or, equivalently, if it is the graph induced by the comparability relation of a partial ordering of the vertices.

**Theorem 1.2** (Dilworth [4]). *Comparability graphs are perfect.*

Another important class of perfect graphs is the class of chordal graphs. Recall that a **chordal graph** (also known as a **triangulated graph**) is a graph that has no induced copies of the cycle  $C_n$  for  $n \geq 4$ . (See Theorem 4.1 for an alternate characterization.)

**Theorem 1.3** (Hajnal–Surányi [7]; Berge [2]). *Chordal graphs are perfect.*

**Interval graphs** (i.e., intersection graphs of families of non-empty convex subsets of a linear order) are also perfect. This can be seen in two ways: because every interval graph is a chordal graph, or because the complement of every interval graph is a comparability graph.

The following result is a typical use of the compactness theorem in graph theory.

**Theorem 1.4.** *Let  $G = (V, E)$  be a graph and let  $k$  be a positive integer.*

- (a)  $\chi(G) \leq k$  if and only if  $\chi(G_X) \leq k$  for every finite  $X \subseteq V$ .
- (b)  $\theta(G) \leq k$  if and only if  $\theta(G_X) \leq k$  for every finite  $X \subseteq V$ .

For perfect graphs, we have a very strong form of this fact.

**Corollary 1.5.** *Let  $G$  be a perfect graph and let  $k$  be a positive integer.*

- (a)  $\chi(G) \leq k$  if and only if  $\chi(G_X) \leq k$  for every  $X \in [V]^{k+1}$ .
- (b)  $\theta(G) \leq k$  if and only if  $\theta(G_X) \leq k$  for every  $X \in [V]^{k+1}$ .

*Proof.* It is enough to prove (a) since (b) is dual. Note that  $\chi(G_X) \leq k$  for every  $X \in [V]^{k+1}$  if and only if  $\omega(G) \leq k$ . By Theorem 1.1,  $\omega(G) \leq k$  if and only if  $\chi(G_X) \leq k$  for every finite  $X \subseteq V$ .  $\square$

It is natural to ask whether the same holds if one replaces  $k$  by an infinite cardinal  $\kappa$  and  $k + 1$  by its cardinal successor  $\kappa^+$ . We will concentrate in the first case,  $\kappa = \aleph_0$  and  $\kappa^+ = \aleph_1$ .

**Definition 1.6.** *Let  $\Gamma$  be a class of graphs. We use  $\mathcal{C}_\chi$  and  $\mathcal{C}_\theta$  denote the following dual statements.*

- ( $\mathcal{C}_\chi$ ) *For every  $G \in \Gamma$ ,  $\chi(G) \leq \aleph_0$  if and only if  $\chi(G_X) \leq \aleph_0$  for every  $X \in [V]^{\aleph_1}$ .*
- ( $\mathcal{C}_\theta$ ) *For every  $G \in \Gamma$ ,  $\theta(G) \leq \aleph_0$  if and only if  $\theta(G_X) \leq \aleph_0$  for every  $X \in [V]^{\aleph_1}$ .*

In 1968, Galvin [6] conjectured that  $\mathcal{C}_\theta$  holds for the class of comparability graphs. In 1981, Rado [13] conjectured that the class of interval graphs has property  $\mathcal{C}_\chi$ . The consistency, relative to the existence of a supercompact cardinal, of Rado's Conjecture was then established by Todorćević [14] in 1983. In [14] and [15], Todorćević shows that large cardinals are indeed necessary to establish the consistency of Rado's Conjecture.

In this paper, we will show that Todorćević's result on the consistency of Rado's Conjecture can be extended to the consistency of  $\mathcal{C}_\chi$  for the class of all chordal graphs.

**Theorem 1.7.** *Each of the following statements implies the next.*

- (a)  $\mathcal{C}_\chi$  holds for the class of  $\sigma$ -treeable graphs.
- (b)  $\mathcal{C}_\chi$  holds for the class of chordal graphs.
- (c)  $\mathcal{C}_\chi$  holds for the class of interval graphs (Rado's Conjecture).

*Furthermore, these statements are all consistent relative to the existence of a supercompact cardinal.*

This theorem will be proved in Section 4 (where we also define  $\sigma$ -treeable graphs). We do not know if any of the implications of Theorem 1.7 are strict since the same technique is used to prove the consistency in all cases.

We will also provide a proof of the following result of Todorćević which shows that Rado's Conjecture is equivalent to the restriction of Galvin's Conjecture to the class of finite-dimensional comparability graphs.

**Theorem 1.8** (Todorćević). *The following are equivalent.*

- (a)  $\mathcal{C}_\chi$  holds for the class of interval graphs (Rado's Conjecture).
- (b)  $\mathcal{C}_\chi$  holds for the class of 2-dimensional comparability graphs.
- (c)  $\mathcal{C}_\theta$  holds for the class of 2-dimensional comparability graphs.
- (d)  $\mathcal{C}_\theta$  holds for the class of finite-dimensional comparability graphs.

The equivalence of (a) and (d) appears without proof in [16, Remark 4.6]. A proof of this theorem will be provided in Section 3 (where we also define  $n$ -dimensional comparability graphs).

While the consistency of Galvin's Conjecture remains open, the above results lead us to the following more general question.

**Question 1.9.** *Is it consistent, relative to large cardinals, that  $\mathcal{C}_\chi$  and, equivalently,  $\mathcal{C}_\theta$  hold for the class of perfect graphs?*

In view of Theorem 1.7, it is natural to ask about  $\mathcal{C}_\theta$  for the class of chordal graphs. It turns out that  $\mathcal{C}_\theta$  is simply true for this class. In fact, property  $\mathcal{C}_\theta$  holds for the broader class of **squarefree** graphs, i.e., graph that do not contain induced copies of the square  $C_4$ . This follows from a result of Wagon.

**Theorem 1.10** (Wagon [18]). *Suppose  $G$  is a squarefree graph such that  $\alpha(G) \leq \aleph_0$ . Then  $\theta(G) > \aleph_0$  if and only if  $G$  contains an induced copy of the comparability graph of a Suslin tree.*

Since Suslin trees have size  $\aleph_1$ , we have the following immediate corollary.

**Corollary 1.11.**  *$\mathcal{C}_\theta$  holds for the class of squarefree graphs, and hence for the class of chordal graphs.*

The techniques used by Wagon suggest that many squarefree graphs are  $\sigma$ -treeable, so there is a chance that the dual of Corollary 1.11 is consistent relative to large cardinals.

**Question 1.12.** *Is it consistent, relative to large cardinals, that  $\mathcal{C}_\chi$  holds for the class of squarefree graphs?*

## 2 Results of Abraham and Todorćević

In this section we summarize some earlier theorems that shed some light on the conjectures of Galvin and Rado. The first due to Abraham and the second due to Todorćević. Abraham's result shows that Galvin's Conjecture holds for the class of comparability graphs without infinite antichains. Todorćević's result gives several equivalent forms of Rado's Conjecture in terms of one-dimensional partition relations for posets.

In 1963, Perles [11] showed that Dilworth's Theorem ( $\alpha(G) = \theta(G)$  for finite comparability graphs) fails for infinite comparability graphs by observing that the cartesian product  $\omega_1 \times \omega_1$  has no infinite antichains but cannot be covered by countably many chains. This example can be generalized as follows.

**Definition 2.1** (Abraham [1]). *A poset  $P$  is of **Perles type** if there is an enumeration  $\langle p_\alpha : \alpha < \omega_1 \rangle$  of  $P$  and a function  $f : \omega_1 \rightarrow \omega_1$  such that  $|f^{-1}(\alpha)| = \aleph_1$  for every  $\alpha < \omega_1$ , and  $\alpha < \beta \wedge f(\alpha) > f(\beta)$  imply that  $p_\alpha$  and  $p_\beta$  are incomparable.*

This definition captures the essential features of  $\omega_1 \times \omega_1$  that were used in Perles's counterexample. Abraham [1] then showed that these are essentially the only counterexamples to Dilworth's Theorem that don't have infinite antichains.

**Theorem 2.2** (Abraham [1]). *Suppose  $P$  is a poset without infinite antichains. Then  $P$  is the union of countably many chains if and only if it does not contain a poset of Perles type.*

Since the posets of Perles type all have size  $\aleph_1$ , it follows immediately that:

**Corollary 2.3.**  *$\mathcal{C}_\theta$  holds for the class comparability graphs without infinite anticliques.*

To state Todorćević's result, it is convenient to introduce some "Hungarian notation" for one-dimensional partitions of posets. If  $\psi$  is a partial order type and  $\kappa$  is a cardinal, we write  $P \rightarrow (\psi)_\kappa^1$  if for every coloring  $c : P \rightarrow \kappa$ , there is a  $Q \subseteq P$  with order type  $\psi$  such that  $c$  is constant on  $Q$ ;  $P \nrightarrow (\psi)_\kappa^1$  denotes the negation of this statement. Generalizing this notation a little, if  $\psi_1, \dots, \psi_k$  are partial order types and  $\kappa$  is a cardinal, we write  $P \rightarrow (\psi_1 \vee \dots \vee \psi_k)_\kappa^1$  if for every coloring  $c : P \rightarrow \kappa$ , there is a  $Q \subseteq P$ , with order type among  $\psi_1, \dots, \psi_k$ , such that  $c$  is constant on  $Q$ ; again  $P \nrightarrow (\psi_1 \vee \dots \vee \psi_k)_\kappa^1$  denotes the negation of this statement.

We will mostly be interested in the negative cases when  $\psi \in \{2, \omega, \omega^*\}$ . Indeed,  $P \nrightarrow (2)_\kappa^1$  simply means that  $P$  is the union of at most  $\kappa$  antichains, i.e.,  $\theta(G_P) \leq \kappa$  where  $G_P$  is the comparability graph of  $P$ . Similarly,  $P \nrightarrow (\omega^*)_ \kappa^1$  (resp.  $P \nrightarrow (\omega)_\kappa^1$ ) means that  $P$  is the union of at most  $\kappa$  well-founded (resp. conversely well-founded) subsets. Finally,  $P \nrightarrow (\omega \vee \omega^*)_ \kappa^1$  means that  $P$  is the union of  $\kappa$  subsets without infinite chains.

**Theorem 2.4** (Todorćević [14]). *The following are equivalent to Rado's Conjecture (i.e.,  $\mathcal{C}_\chi$  holds for the class of interval graphs).*

- (a) *For every tree  $T$ ,  $T \nrightarrow (2)_\omega^1$  if and only if  $U \nrightarrow (2)_\omega^1$  for every  $U \in [T]^{\aleph_1}$ .*
- (b) *For every poset  $P$ ,  $P \nrightarrow (\omega)_\omega^1$  if and only if  $Q \nrightarrow (\omega)_\omega^1$  for every  $Q \in [P]^{\aleph_1}$ .*
- (c) *For every poset  $P$ ,  $P \nrightarrow (\omega^*)_ \omega^1$  if and only if  $Q \nrightarrow (\omega^*)_ \omega^1$  for every  $Q \in [P]^{\aleph_1}$ .*
- (d) *For every poset  $P$ ,  $P \nrightarrow (\omega \vee \omega^*)_ \omega^1$  if and only if  $Q \nrightarrow (\omega \vee \omega^*)_ \omega^1$  for every  $Q \in [P]^{\aleph_1}$ .*

*Proof.* The equivalence of Rado's Conjecture with (a) and (b) is [14, Theorem 6]; (c) is equivalent to (b), by duality; (d) follows from the combination of (b) and (c); (d) implies (a) since  $T \nrightarrow (2)_\omega^1$ ,  $T \nrightarrow (\omega)_\omega^1$ , and  $T \nrightarrow (\omega \vee \omega^*)_\omega^1$  are all equivalent for a tree  $T$ .  $\square$

### 3 Finite-Dimensional Comparability Graphs

A  **$n$ -dimensional poset** is a poset  $P$  whose order relation is the intersection of  $n$  linear orders, i.e., if there are linear orders  $\leq_1, \dots, \leq_n$  on the points of  $P$  such that  $x \leq_P y \Leftrightarrow x \leq_1 y \wedge \dots \wedge x \leq_n y$ . It turns out that the dimension of a poset is an invariant of its comparability graph.

**Theorem 3.1** (Trotter–Moore–Sumner [17]). *If the graph  $G$  has an  $n$ -dimensional transitive orientation, then every transitive orientation of  $G$  is  $n$ -dimensional.*

Thus, it makes sense to say that  $G$  is a  **$n$ -dimensional comparability graph** if  $G$  has an  $n$ -dimensional transitive orientation.

The class of 2-dimensional comparability graphs is especially interesting since it is self-dual.

**Theorem 3.2** (Pnueli–Lempel–Even [12]). *A graph  $G$  is a 2-dimensional comparability graph if and only if  $G$  and its complement  $\overline{G}$  are both comparability graphs.*

This last result immediately implies the equivalence of (b) and (c) in Theorem 1.8. The next result shows that (c) implies (d) in Theorem 1.8. This establishes the equivalence of the last three statements of Theorem 1.8.

**Theorem 3.3.** *If  $\mathcal{C}_\theta$  holds for the class of 2-dimensional comparability graphs, then  $\mathcal{C}_\theta$  holds for the class of finite-dimensional comparability graphs.*

*Proof.* We proceed by induction on dimension. Suppose that  $\mathcal{C}_\theta$  holds for every  $n$ -dimensional comparability graph. Let  $P = (V, \leq)$  be a  $(n+1)$ -dimensional poset. Then there are a  $n$ -dimensional partial order  $\leq_0$  and a linear order  $\leq_1$  on  $V$  such that  $u \leq v \Leftrightarrow u \leq_0 v \wedge u \leq_1 v$ . Write  $P_0 = (V, \leq_0)$  and  $P_1 = (V, \leq_1)$ . If every  $U \in [V]^{\aleph_1}$  is the union of countably many  $\leq$ -chains, then it is also the union of countably many  $\leq_0$ -chains. Therefore, by the induction hypothesis,  $V$  is the union of countably many  $\leq_0$ -chains, say  $V = \bigcup_{n=0}^\infty C_n$  where each  $C_n$  is a  $\leq_0$ -chain. Now the restriction of  $\leq$  to  $C_n$  is 2-dimensional as  $\leq_0$  and  $\leq_1$  are both linear orders on  $C_n$ . Also, by hypothesis, every  $D \in [C_n]^{\aleph_1}$  is the union of countably many  $\leq$ -chains.

Since  $\mathcal{C}_\theta$  holds for 2-dimensional comparability graphs, each  $C_n$  is itself the union of countably many  $\leq$ -chains. Gathering these smaller chains together, we find that  $V$  is the union of countably many  $\leq$ -chains.  $\square$

For the last equivalent form of Theorem 1.8, we appeal to the results of Todorćević and Abraham from the previous section.

**Theorem 3.4.** *Rado's Conjecture implies that  $\mathcal{C}_\theta$  holds for the class of 2-dimensional comparability graphs.*

*Proof.* Let  $P$  be a 2-dimensional poset and let  $P'$  be a poset whose comparability graph is the complement of that of  $P$ . Assume that every  $Q \in [P]^{\aleph_1}$  is the union of countably many chains or, dually, every  $Q' \in [P']^{\aleph_1}$  is the union of countably many antichains. Then we have  $Q' \nrightarrow (\omega \vee \omega^*)_\omega^1$  (indeed  $Q' \nrightarrow (2)_\omega^1$ ) for every  $Q' \in [P']^{\aleph_1}$ . Hence,  $P' \nrightarrow (\omega \vee \omega^*)_\omega^1$ , by Theorem 2.4. Thus,  $P'$  is the union of countably many sets each of which has no infinite chains. It follows by duality that  $P = \bigcup_{n=0}^{\infty} R_n$  where each  $R_n$  has no infinite antichains. Now, every  $Q \in [R_n]^{\aleph_1} \subseteq [P]^{\aleph_1}$  is the union of countably many chains. It follows from Corollary 2.3, that each  $R_n$  is the union of countably many chains. Gathering these chains together, we see that  $P$  is the union of countably many chains.  $\square$

This shows that Rado's Conjecture implies Galvin's Conjecture for 2-dimensional posets. For the converse, we show that Galvin's Conjecture for 2-dimensional posets implies the first equivalent form of Rado's Conjecture in Theorem 2.4.

**Theorem 3.5.** *If  $\mathcal{C}_\chi$  holds for the class of 2-dimensional comparability graphs then, for every tree  $T$ , we have  $T \nrightarrow (2)_\omega^1$  if and only if  $U \nrightarrow (2)_\omega^1$  for every  $U \in [T]^{\aleph_1}$ .*

*Proof.* It suffices to observe that every tree  $T$  is a 2-dimensional poset, which can be seen by lexicographically ordering  $T$  in two opposite ways.  $\square$

## 4 Interval, Chordal, and $\sigma$ -Treeable Graphs

The following characterization of chordal graphs is due to Fulkerson and Gross [5] in the finite case; the infinite case follows by a simple application of the Compactness Theorem. An orientation  $\vec{E}$  of  $G = (V, E)$  is said to be a **simplicial orientation** if it is acyclic and  $S_v = \{u \in V : u \vec{E} v\}$  is a clique in  $G$  for every  $v \in V$ .

**Theorem 4.1** (Fulkerson–Gross [5]). *A graph  $G = (V, E)$  is chordal if and only if it has a simplicial orientation.*



With this result, it is easy to show that interval graphs are chordal.

**Corollary 4.2.** *Every interval graph is chordal.*

*Proof.* Let  $G = (V, E)$  be an interval graph as witnessed by the family of intervals  $\langle I_v : v \in V \rangle$  of a linear order  $L$ . Define  $u \vec{E} v$  iff  $I_u \cap I_v$  is a nonempty initial subinterval of  $I_v$ . (If some of the intervals are equal, break ties using a linear ordering of  $V$ .) It is easy to check that  $\vec{E}$  is a simplicial orientation of  $V$ .  $\square$

It follows immediately that (b) implies (c) in Theorem 1.7.

Before we define the class of  $\sigma$ -treeable graphs, let us make an observation to motivate the definition. A poset  $R = (V, \triangleleft)$  is **ramified** if every initial interval  $R[\triangleleft v]$  is linearly ordered by  $\triangleleft$  for each  $v \in V$ . Thus a tree is simply a well-founded ramified poset.

**Theorem 4.3.** *If  $G = (V, E)$  is a chordal graph, then the transitive closure of any simplicial orientation of  $G$  is a ramified ordering of  $V$ .*

*Proof.* Let  $\triangleleft$  be the transitive closure of a simplicial orientation  $\vec{E}$  of  $G$ . For  $v \in V$ , let  $S_v = \{w \in V : w \vec{E} v\}$ . Then define  $S_v^0 = \{v\}$  and  $S_v^{n+1} = \bigcup \{S_w : w \in S_v^n\}$ . Note that  $S_v^1 = S_v$  and  $w \trianglelefteq v$  iff  $w \in \bigcup_{n=0}^{\infty} S_v^n$ .

We want to show that if  $u, v \trianglelefteq w$  then  $u \trianglelefteq v$  or  $u \triangleright v$ . We proceed by induction on  $m$  where  $u \in S_w^m$ .

For  $m = 0$ , we have  $u = w$  and hence  $u \triangleright v$ .

For  $m = 1$ , let  $v = v_0 \vec{E} v_1 \vec{E} \dots \vec{E} v_n = w$  witness that  $v \trianglelefteq w$ . Let  $p = \min \{i : u \triangleleft v_i\}$ . Note that  $u \vec{E} v_i$  for  $i = p, \dots, n$ . (This is clear for  $i = n$  since  $u \vec{E} w$  by definition of  $S_w$ . Suppose that  $u \vec{E} v_{i+1}$  and  $i \geq p$ , then  $u, v_i \in S_{v_{i+1}}$ , which means that  $u \vec{E} v_i$  since  $S_{v_{i+1}}$  is a clique and  $u \triangleleft v_i$ .) If  $p = 0$  then it follows immediately that  $u \triangleleft v_0 = v$ . If  $p > 0$ , then note that  $v_{p-1} \vec{E} u$  or  $v_{p-1} = u$  since  $u, v_{p-1} \in S_{u_p}$ ,  $S_{u_p}$  is a clique, and  $u \not\vec{E} v_{p-1}$ . Therefore,  $v = v_0 \trianglelefteq v_{p-1} \triangleleft u$ .

For  $m > 1$ , note that  $u \in S_x$  for some  $x \in S_w^{m-1}$ . By the induction hypothesis, either  $x \triangleleft v$ ,  $x = v$ , or  $x \triangleright v$ . If  $x \trianglelefteq v$ , then  $u \triangleleft v$  by transitivity of  $\triangleleft$ . If  $x \triangleright v$ , then the result follows from the case  $m = 1$ , since  $u \in S_x = S_x^1$ .  $\square$

A graph  $G = (V, E)$  is  **$\sigma$ -treeable** if it is contained in the comparability graph of a ramified ordering  $\triangleleft$  of  $V$  which has the additional property that  $|V[\triangleleft v]| \leq \aleph_0$  for every  $v \in V$ . The next lemma will perhaps clarify our choice of terminology.

**Lemma 4.4.** *If  $G = (V, E)$  is  $\sigma$ -treeable, as witnessed by the ramified ordering  $\triangleleft$  of  $V$ , then there is a partition  $V = \bigcup_{n=0}^{\infty} V_n$  such that the restriction of  $\triangleleft$  to each  $V_n$  is a tree of height at most  $\omega_1$ .*

*Proof.* (Due to Galvin, cf. [14].) Fix a well-ordering  $\prec$  of  $V$ . For each  $v \in V$ , let  $f_v : V[\trianglelefteq v] \rightarrow \omega$  be an injection. Define,  $f : V \rightarrow \omega$  by  $f(u) = f_v(u)$  where  $v$  is the  $\prec$ -first element of  $V$  such that  $u \trianglelefteq v$ . We claim that the restriction of  $\triangleleft$  to each  $V_n = f^{-1}(n)$  is well-founded.

Suppose that  $u_0 \triangleright u_1 \triangleright \dots$  is a descending sequence of elements of  $V_n$ . Let  $v_i$  be the  $\prec$ -first  $v \in V$  with  $u_i \trianglelefteq v$ . Note that  $f_{v_i}(u_i) = f(u_i) = n$  for each  $i < \omega$ . Note also that  $v_0 \succeq v_1 \succeq \dots$ . Since  $\prec$  is a well-ordering, there are  $v$  and  $k$  such that  $v_i = v$  for  $i \geq k$ . Since  $f_v$  is an injection we have  $u_i = f_v^{-1}(n)$  for  $i \geq k$ . Thus  $u_0 \triangleright u_1 \triangleright \dots$  is eventually constant, which shows that  $\triangleleft$  is well-founded on  $V_n$ .  $\square$

If  $G = (V, E)$  is any graph such that  $\chi(G_X) \leq \aleph_0$  for every  $X \in [V]^{\aleph_1}$ , then we certainly have  $\omega(G) \leq \aleph_0$ . If, moreover,  $G$  is chordal and  $\vec{E}$  is a simplicial orientation of  $G$ , then  $|S_v| \leq \aleph_0$  for each  $v \in V$ . It then follows that  $|V[\triangleleft v]| \leq \aleph_0$  for each  $v \in V$  where  $\triangleleft$  is the transitive closure of  $\vec{E}$ . Therefore, every chordal graph such that  $\chi(G_X) \leq \aleph_0$  for every  $X \in [V]^{\aleph_1}$ , is  $\sigma$ -treeable. This shows that (a) implies (b) in Theorem 1.7.

To complete the proof of Theorem 1.7, it remains to prove the consistency of  $\mathcal{C}_\chi$  for the class of  $\sigma$ -treeable graphs, relative to the existence of a supercompact cardinal. Rather than giving a forcing proof the consistency of  $\mathcal{C}_\chi$  for  $\sigma$ -treeable graphs, as in [14], we will use the Global Game Reflection Principle ( $\text{GRP}^+$ ) of [8]. Let  $\mathcal{S} \subseteq (A \times B)^{<\omega_1}$  be a tree and let  $[\mathcal{S}] = \{s \in (A \times B)^{\omega_1} : (\forall \alpha < \omega_1)(s \restriction \alpha \in \mathcal{S})\}$ . Consider a two player game  $\mathbb{G}(\mathcal{S})$  of length  $\omega_1$  where in each round  $\alpha < \omega_1$ , Player I plays  $a_\alpha \in A$ , Player II responds with  $b_\alpha \in B$ , and Player II wins if  $\langle (a_\alpha, b_\alpha) : \alpha < \omega_1 \rangle \in [\mathcal{S}]$ . If  $X \subseteq A$ , then the restricted game  $\mathbb{G}(\mathcal{S}|X)$  is defined similarly except that Player I can only play elements of  $X$ .

( $\text{GRP}^+$ ) If  $\mathcal{S} \subseteq (A \times B)^{<\omega_1}$  is a tree,  $\mathcal{C} \subseteq [A]^{\aleph_1}$  is an  $\omega_1$ -club, and Player II has a winning strategy in the restricted game  $\mathbb{G}(\mathcal{S}|X)$  for every  $X \in \mathcal{C}$ , then Player II has a winning strategy in the unrestricted game  $\mathbb{G}(\mathcal{S})$ .

It is known that this principle has considerable large cardinal strength, but no more than a supercompact cardinal. In fact, the consistency of  $\text{GRP}^+$  can be obtained by the Lévy collapse of a supercompact cardinal to  $\aleph_2$ .

**Theorem 4.5** (König [8]). *If  $\kappa$  is supercompact, then  $\text{Coll}(\aleph_1, <\kappa) \Vdash \text{GRP}^+$ .*

It is also observed in [8] that Rado's Conjecture follows from  $\text{GRP}^+$ . Here we prove the more general result that  $\text{GRP}^+$  implies  $\mathcal{C}_\chi$  for  $\sigma$ -treeable graphs.

**Theorem 4.6.**  $\text{GRP}^+$  implies that  $\mathcal{C}_\chi$  holds for  $\sigma$ -treeable graphs.

*Proof.* Let  $G = (V, E)$  be a  $\sigma$ -treeable graph as witnessed by the partial ordering  $\triangleleft$  of  $V$ . By Lemma 4.4, we may assume that  $(V, \triangleleft)$  is a tree of height at most  $\omega_1$ .

Consider the game  $\mathbb{G}_\chi(G)$  of length  $\omega_1$  where, in each round  $\alpha < \omega_1$ , Player I plays  $v_\alpha \in V$ , Player II responds with  $c_\alpha \in \omega$ , and Player II wins iff  $v_\alpha E v_\beta \Rightarrow c_\alpha \neq c_\beta$  for all  $\alpha < \beta < \omega_1$ . The fact that  $\chi(G_W) \leq \aleph_0$  for every  $W \in [V]^{\aleph_1}$  clearly implies that Player II has a winning strategy for the restricted game  $\mathbb{G}_\chi(G|W)$ . Therefore, by  $\text{GRP}^+$ , Player II has a winning strategy in the unrestricted game  $\mathbb{G}_\chi(G)$ .

Define the coloring  $c : V \rightarrow \omega$  as follows. Suppose that  $v \in V$  has height  $\eta < \omega_1$  and let  $\langle v_\alpha : \alpha \leq \eta \rangle$  enumerate the branch  $V[\leq v]$  in  $\triangleleft$ -order (so  $v_\eta = v$ ). Consider the sequence  $\langle v_\alpha : \alpha \leq \eta \rangle$  as a sequence of moves for Player I in the game  $\mathbb{G}_\chi(G)$  and let  $\langle c_\alpha : \alpha \leq \eta \rangle$  be the sequence of Player II responses according to her winning strategy. Then set  $c(v) = c_\eta$ . Note that  $c_\alpha = c(v_\alpha)$  for every  $\alpha \leq \eta$ . Since Player II was using her winning strategy in this play, it follows that  $v_\alpha E v_\eta \Rightarrow c(v_\alpha) = c_\alpha \neq c_\eta = c(v_\eta)$ . Therefore,  $c : V \rightarrow \omega$  is a proper coloring of  $G$  and hence  $\chi(G) \leq \aleph_0$ .  $\square$

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